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Dressing methods for geometric nets: I. Conjugate nets

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Abstract. The formalism of multicomponent KP hierarchies is applied to deriving efficient dressing methods for conjugate nets. The notion of the Cauchy propagator is used for characterizing these nets in terms of spectral data. Explicit examples in dimensions $N = 2$ and 3 are given. In particular, periodic nets and Cartesian nets with a Gaussian localized deformation are exhibited.

(Some figures in this article appear in black and white in the printed version.)

1. Introduction

In recent years [5, 22, 23] it has been found that the theory of integrable systems is a useful tool to study geometric nets $\boldsymbol{x} = \boldsymbol{x}(u_1, \dots, u_M)$ of conjugate type in Euclidean space [1, 2]. They are characterized by the Laplace equations

$$\frac{\partial^2 \boldsymbol{x}}{\partial u_i \partial u_j} = \frac{\partial \ln H_i}{\partial u_j}\frac{\partial \boldsymbol{x}}{\partial u_i} + \frac{\partial \ln H_j}{\partial u_i}\frac{\partial \boldsymbol{x}}{\partial u_j} \qquad i, j = 1, \dots, M \qquad i \neq j \qquad (1)$$

where H_i are the so-called *Lamé coefficients*. The compatibility conditions for the above equations provide a relevant integrable model: the Darboux system

$$\frac{\partial \beta_{ik}}{\partial u_k} = \beta_{ik}\beta_{kj} \qquad i, j \text{ and } k \text{ different} \qquad (2)$$

for the *rotation coefficients* β_{ij}

$$\beta_{ji} := \frac{1}{H_j}\frac{\partial H_i}{\partial u_j}.$$

Each solution β_{ij} of (2) determines a family of parallel conjugate nets $\boldsymbol{x}$ given by the solutions of

$$\frac{\partial \boldsymbol{x}}{\partial u_i} = H_i \boldsymbol{X}_i.$$

Here $\boldsymbol{X}_i$ stands for the *re-normalized tangent vectors* of the net defined by

$$\frac{\partial \boldsymbol{X}_i}{\partial u_j} = \beta_{ij}\boldsymbol{X}_j \qquad i \neq j.$$

One of the main examples of conjugate nets is the class of orthogonal systems of curvilinear coordinates in Euclidean space [3, 16] or, equivalently, the set of flat diagonal metrics

$$\mathrm{d}s^2 = \sum_{i=1}^{M} H_i^2 (\mathrm{d}u_i)^2.$$

In this case the rotation coefficients and the tangent vectors are constrained by additional differential equations [16,22]. A particularly important type of orthogonal net, the ∂-invariant Egorov net [8, 10], is defined by supplementing (2) with the conditions

$$\beta_{ij} = \beta_{ji} \qquad \partial\beta_{ij} = \partial H_i = 0$$

where $\partial := \sum_{i=1}^{M} \frac{\partial}{\partial u_i}$. The analysis of the integrability properties of these nets has revealed their importance in the problem of the characterization of two-dimensional topological field theories (TFT) and related mathematical structures as, for example, solutions of the Witten–Dijkgraff–Verlinde–Verlinde (WDVV) equation [4, 21], Frobenius manifolds and systems of hydrodynamic type. In this context, given a ∂-invariant Egorov net, a prominent role is played by the *deformed flat coordinates* $\theta_\alpha(z, \boldsymbol{x})$ of its associated Frobenius manifold [8], where x^α are the flat coordinates of the Egorov net and z is the spectral parameter of the underlying integrability theory.

In [5,18] the formalism of multicomponent KP hierarchies was used for studying conjugate and orthogonal nets and it was proved that β_{ij}, H_i and $\boldsymbol{X}_i$ can be written in terms of τ-functions and wavefunctions (Baker–Akhiezer functions). Moreover, dressing transformations for these geometric objects were given. However, no similar results for the *net function* $\boldsymbol{x}(u_1, \ldots, u_M)$ were derived. In this sense we note that, in addition to its intrinsic geometric value, the characterization of the net function as an analytic object in the framework of the KP theory is a basic step in order to design an efficient dressing method for solving (1).

This paper deals with the description of the net function in the KP theory and the formulation of the corresponding dressing method. These questions are considered within the context of the Grassmannian formalism of the KP hierarchies [19]. A basic ingredient of our study is the use of a Cauchy propagator $\Psi(z, z')$ [13,20]

$$\frac{\partial\Psi}{\partial\bar{z}}(z, z') = \pi\delta(z - z')$$

satisfying appropriate boundary conditions. We note [14] that the notion of the Cauchy propagator is useful for calculating the Virasoro action on the algebraic-geometric data of KP solutions as well as the action of vertex operators on the corresponding τ-functions. To describe our main results, we recall [5] that given a KP wavefunction $\psi(z)$ and its adjoint function $\psi^*(z)$, the corresponding solution β_{ij} of (2) admits tangent vectors and Lamé coefficients given by $(\boldsymbol{X}_i)_j := \boldsymbol{X}_{ij}$ and $H_i = \boldsymbol{H}_{li}$, $(l = 1, \ldots, N)$, where

$$\boldsymbol{X}(\boldsymbol{u}) := \int_{\mathbb{C}} \psi(z, \boldsymbol{u}) N(z)\, \mathrm{d}^2 z$$

and

$$\boldsymbol{H}(\boldsymbol{u}) := \int_{\mathbb{C}} M(z)\psi^*(z, \boldsymbol{u})\, \mathrm{d}^2 z.$$

Here $N(z)$ and $M(z)$ are appropriate $N \times N$ matrix distributions. In this paper it is proved that the corresponding net functions are the rows of

$$\boldsymbol{x}(\boldsymbol{u}) := \int_{\mathbb{C}\times\mathbb{C}} M(z')\Psi(z, z')N(z)\, \mathrm{d}^2 z\, \mathrm{d}^2 z' + \boldsymbol{x}_0$$

where $\boldsymbol{x}_0$ is an arbitrary constant matrix. From this characterization we are able to formulate a dressing method for Cauchy propagators based on the use of $\bar{\partial}$-equations. Furthermore, we obtain a closed formula for the dressing transformations corresponding to the separable case of the $\bar{\partial}$-equation involved. The outcome is a dressing method for conjugate nets which constitutes a *spectral* version of the classical fundamental transformation [7, 11, 12, 15]. Several interesting classes of explicit conjugate nets obtained by this method are exhibited. In particular, we concentrate on periodic and Hermite nets. The former are trigonometric nets which exhibit a periodic behaviour, while the latter represent a Cartesian net with a localized dislocation.

We must note that in [7] an alternative $\bar{\partial}$ approach to the geometrical transformations of conjugate nets and quadrilateral lattices was given. It should also be stressed that a detailed study of the Cauchy propagator for quadrilateral lattices and a general $\bar{\partial}$ reduction theory which includes, as distinguished examples, the continuous and discrete orthogonal, symmetric, d-invariant and Egorov cases and the construction scheme for the separable solutions of the above geometric objects can be found in [6].

The second paper of this series will be concerned with the theories of orthogonal and ∂-invariant Egorov nets [3, 8, 9, 16]. In particular, the following questions will be dealt with:

(i) The reductions of the dressing method.
(ii) The generation of relevant classes of orthogonal nets.
(iii) The use of Cauchy propagators and dressing methods for analysing deformed flat coordinates θ_α in the theory of ∂-invariant Egorov nets and the free-energy functions of the corresponding TFT.

2. KP theory of conjugate nets

2.1. Multicomponent KP hierarchies

The N-component KP hierarchy can be introduced from the consideration of a certain family of flows in an infinite-dimensional Grassmannian [19]. To describe this process we denote by $D(r)$ and $\gamma(r)$ the disc $\{z \in \mathbb{C} : |z| \leqslant r\}$ and its boundary $\{z \in \mathbb{C} : |z| = r\}$, respectively, and introduce the set $H_{\gamma(r)}$ of Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

with coefficients a_n in the algebra $M_N(\mathbb{C})$ of $N \times N$ complex matrices, which converge on the circle $\gamma(r)$. Next, two different Grassmannians $\mathrm{Gr}_{\gamma(r)}$ and $\mathrm{Gr}^*_{\gamma(r)}$ are required.

Definition 1. *The elements of $Gr_{\gamma(r)}$ are the subsets W of $H_{\gamma(r)}$ such that:*

(1) W is a $M_N(\mathbb{C})$ left-module.
(2) The projection operator $P_+ : W \longrightarrow H^+_{\gamma(r)}$ from W into $H^+_{\gamma(r)} = \{w \in H_{\gamma(r)} : w = \sum_{n=0}^{\infty} a_n z^n\}$ is a bijective map.

Similarly, $Gr^*_{\gamma(r)}$ is given by the subsets V of H_γ such that:

(1) V is a $M_N(\mathbb{C})$ right-module.*
(2) The projection operator $P_+ : V \longrightarrow H^+_{\gamma(r)}$ is a bijective map.*

There is a map

$$\mathrm{Gr}_{\gamma(r)} \xrightarrow{*} \mathrm{Gr}^*_{\gamma(r)} \qquad W \mapsto W^*$$

which associates to each $W \in \mathrm{Gr}_{\gamma(r)}$ the element $W^* \in \mathrm{Gr}^*_{\gamma(r)}$ given by those $v \in H_{\gamma(r)}$ such that

$$\int_{\gamma(r)} w(z)v(z)\,\mathrm{d}z = 0 \qquad \forall w \in W. \tag{3}$$

One of the most useful ways for characterizing elements in the Grassmannians is the $\bar\partial$-method. It starts from an appropriate $N \times N$ matrix distribution $R(z, z')$ with support in $D(r) \times D(r)$, and then determines $W \in \mathrm{Gr}_{\gamma(r)}$ as the set of restrictions to $\gamma(r)$ of the solutions $w = w(z)$ of the equation

$$\frac{\partial w}{\partial \bar z}(z) = \int_{D(r)} w(z')R(z', z)\,\mathrm{d}^2 z'.$$

In this case the corresponding element $W^* \in \mathrm{Gr}^*_{\gamma(r)}$ is determined by the solutions of

$$\frac{\partial v}{\partial \bar z}(z) = -\int_{D(r)} R(z, z')v(z')\,\mathrm{d}^2 z'.$$

The KP flows on the Grassmannians are implemented by multiplication operators as follows.

Definition 2. *Given $W \in Gr_{\gamma(r)}$ and $V \in Gr^*_{\gamma(r)}$ their KP flows are defined by*

$$W(\boldsymbol{u}) = W\psi_0^{-1}(z, \boldsymbol{u})$$
$$V(\boldsymbol{u}) = \psi_0(z, \boldsymbol{u})V$$

where $\boldsymbol{u} := (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N)$ denotes N infinite sequences $\boldsymbol{u}_i := (u_{i,1}, u_{i,2}, \ldots) \in \mathbb{C}^\infty$, and

$$\psi_0(z, \boldsymbol{u}) := \exp\left[\sum_{n\geqslant 1} z^n\left(\sum_{i=1}^{N} u_{i,n}E_i\right)\right] \qquad (E_i)_{jk} = \delta_{ij}\delta_{ik}.$$

In order to prevent $\psi_0(z, \boldsymbol{u})$ from having singularities as a function of z in $D(r)$, we will assume henceforth that the domain of each variable $\boldsymbol{u}_i$ is

$$\mathcal{U}(r) = \left\{\boldsymbol{a} = (a_1, a_2, \ldots) \in \mathbb{C}^\infty : \sum_{n\geqslant 1} z^n a_n \text{ converges for } z \in D(r)\right\}.$$

From this assumption it follows that the subsets of the Grassmannians characterized by the $\bar\partial$-method are invariant under the action of the KP flows. Thus, if $W \in \mathrm{Gr}_{\gamma(r)}$ is determined by a kernel $R(z, z')$, then $W(\boldsymbol{u})$ is in turn determined by

$$R(z, z', \boldsymbol{u}) := \psi_0(z, \boldsymbol{u})R(z, z')\psi_0(z', \boldsymbol{u})^{-1}.$$

Definition 3. *Given $W \in Gr_{\gamma(r)}$ we define its associated KP wavefunction (Baker–Akhiezer function) as the unique function $\psi = \psi(z, \boldsymbol{u})$ such that it admits a convergent expansion of the form*

$$\psi(z, \boldsymbol{u}) = \chi(z, \boldsymbol{u})\psi_0(z, \boldsymbol{u}) \qquad \chi(z, \boldsymbol{u}) = I_N + \sum_{n\geqslant 1} \frac{a_n(\boldsymbol{u})}{z^n}$$
$$\boldsymbol{u} \in \mathcal{U}(r)^N \qquad z \in \gamma(r).$$

Furthermore, we define the adjoint KP wavefunction associated to W as the unique function $\psi^* = \psi^*(z, \boldsymbol{u})$, with an expansion of the form

$$\psi^*(z, \boldsymbol{u}) = \psi_0(z, \boldsymbol{u})^{-1}\chi^*(z, \boldsymbol{u}) \qquad \chi^*(z, \boldsymbol{u}) = I_N + \sum_{n\geqslant 1} \frac{a_n^*(\boldsymbol{u})}{z^n}$$
$$\boldsymbol{u} \in \mathcal{U}(r)^N \qquad z \in \gamma(r).$$

In the above definition we denote by $I_N := \sum_{i=1}^N E_i$ the identity matrix in $M_N(\mathbb{C})$. We notice that for all $\boldsymbol{u} \in \mathcal{U}(r)^N$, both $\chi(z,\boldsymbol{u})$ and $\chi^*(z,\boldsymbol{u})$ are analytic functions of z on the domain $\mathbb{C}\backslash D(r)$.

From (3) we deduce that these wavefunctions satisfy the bilinear identity

$$\int_{\gamma(r)} \psi(z,\boldsymbol{u})\psi^*(z,\boldsymbol{u}')\,\mathrm{d}z = 0. \tag{4}$$

A handful of relations can be derived by using this identity, in particular if we set $\boldsymbol{u} = \boldsymbol{u}'$ we get $a_1^* = -a_1$.

A very helpful notion for deriving differential identities in KP hierarchies is the concept of *normalization*.

Definition 4. *Let $w = w(z,\boldsymbol{u})$ be a function such that its restriction to $\gamma(r)$ satisfies either*

$$w(\cdot,\boldsymbol{u}) \in W(\boldsymbol{u}) \qquad \forall \boldsymbol{u} \in \mathcal{U}(r)^N$$

or

$$w(\cdot,\boldsymbol{u}) \in W^*(\boldsymbol{u}) \qquad \forall \boldsymbol{u} \in \mathcal{U}(r)^N.$$

Then, its normalization is defined by

$$\mathfrak{N}[w(z,\boldsymbol{u})] := P_+ w(z,\boldsymbol{u}).$$

From definition 1 it is clear that $w(\cdot,\boldsymbol{u}) \in W(\boldsymbol{u})$ or $w(\cdot,\boldsymbol{u}) \in W^*(\boldsymbol{u})$ are uniquely determined by their normalization. In particular $\chi(z,\boldsymbol{u})$ and $\chi^*(z,\boldsymbol{u})$ of definition 3 are the functions with unit normalization in $W(\boldsymbol{u})$ and $W^*(\boldsymbol{u})$, respectively. Thus, by taking into account that

$$\mathfrak{N}\left[\frac{\partial\psi}{\partial u_{i,n}}\psi_0^{-1}\right] = z^n E_i + \mathcal{O}(z^{n-1})$$

$$\mathfrak{N}[(\partial^n\psi)\psi_0^{-1}] = z^n + \mathcal{O}(z^{n-1}) \qquad \partial := \sum_{i=1}^N \frac{\partial}{\partial u_{i,1}}$$

and by identifying elements with the same normalization, we get the KP hierarchy of linear equations

$$\frac{\partial\psi}{\partial u_{i,n}} = P_{i,n}(\boldsymbol{u},\partial)\psi \qquad i = 1,\ldots,N, \qquad n \geqslant 1 \tag{5}$$

where $P_{i,n}(\boldsymbol{u},\partial)$ is a family of linear differential operators in ∂.

The simplest members of the hierarchy (5) are

$$\frac{\partial\psi}{\partial u_{i,1}} = E_i\partial\psi + [a_1, E_i]\psi \qquad i = 1,\ldots,N.$$

Thus, we get

$$\frac{\partial\boldsymbol{\psi}_i}{\partial u_k} = \beta_{ik}\boldsymbol{\psi}_k \qquad i \neq k \tag{6}$$

with

$$\boldsymbol{\psi}_i := (\psi_{i1},\ldots,\psi_{iN}) \qquad u_k := u_{k,1} \qquad \beta = a_1.$$

In a similar fashion, one finds

$$\frac{\partial\boldsymbol{\psi}_j^*}{\partial u_k} = \boldsymbol{\psi}_k^*\beta_{kj} \qquad j \neq k \tag{7}$$

where

$$\psi_i^* := \begin{pmatrix} \psi_{1i}^* \\ \vdots \\ \psi_{Ni}^* \end{pmatrix}.$$

The compatibility of either (6) or (7) implies the Darboux system of equations for a conjugate net (2). Moreover, (6) and (7) show that for a given set of rotation coefficients β_{ij} there is an associated family of conjugate nets with tangent vectors and Lamé coefficients given by $(\boldsymbol{X}_i)_j := \boldsymbol{X}_{ij}$ and $H_i = \boldsymbol{H}_{li}$, $(l = 1, \dots, N)$, where

$$\boldsymbol{X}(\boldsymbol{u}) := \int_{\mathbb{C}} \psi(z, \boldsymbol{u}) N(z)\, \mathrm{d}^2 z$$

and

$$\boldsymbol{H}(\boldsymbol{u}) := \int_{\mathbb{C}} M(z) \psi^*(z, \boldsymbol{u})\, \mathrm{d}^2 z.$$

Here $N(z)$ and $M(z)$ are appropriate $N \times N$ matrix distributions.

2.2. The Cauchy propagator

Definition 5. *Let W be an element of $Gr_{\gamma(r)}$, we define its associated Cauchy propagator as the Green function $\Psi = \Psi(z, z', \boldsymbol{u})$ of the $\bar{\partial}$-operator*

$$\frac{\partial \Psi}{\partial \bar{z}}(z, z', \boldsymbol{u}) = \pi \delta(z - z') \qquad z, z' \in \mathbb{C} \backslash D(r) \qquad \boldsymbol{u} \in \mathcal{U}(\infty)^N$$

satisfying the following boundary conditions:

(1) For every fixed $\boldsymbol{u} \in \mathcal{U}(\infty)^N$ and $z' \in \mathbb{C}\backslash D(r)$ the restriction of Ψ to $\gamma(r)$, as a function of z, is an element of W.

(2) As $z \longrightarrow \infty$

$$\Psi(z, z', \boldsymbol{u}) = \mathcal{O}\left(\frac{1}{z}\right) \psi_0(z, \boldsymbol{u}).$$

Observe that there cannot be two different Cauchy propagators associated to a given $W \in \mathrm{Gr}_{\gamma(r)}$. Indeed, if there were two, let us say Ψ_1 and Ψ_2, then $\Lambda := (\Psi_1 - \Psi_2)\psi_0^{-1}$ would be an analytic function of z on $\mathbb{C}\backslash D(r)$, such that

$$\lim_{z \longrightarrow \infty} \Lambda(z, z', \boldsymbol{u}) = 0.$$

Thus, its restriction to $\gamma(r)$, as a function of z, would satisfy

$$\mathfrak{N}[\Lambda(z, z', \boldsymbol{u})] = 0.$$

Therefore, as this restriction belongs to $W(\boldsymbol{u})$, we conclude that $\Lambda(z, z', \boldsymbol{u})$ vanishes, which proves the uniqueness of the Cauchy propagator.

The next theorem shows how the Cauchy propagator can be expressed in terms of the KP wavefunctions. The following notational convention is used:

$$[z] = ([z]_1, \dots, [z]_N) \qquad [z]_i = \left(\frac{1}{z}, \dots, \frac{1}{nz^n}, \dots\right).$$

Theorem 1. *The Cauchy propagator associated to an element W of $Gr_{\gamma(r)}$ can be written in terms of the KP wavefunctions ψ and ψ^* as*

$$\Psi(z, z', \boldsymbol{u}) = \begin{cases} -\dfrac{1}{z'} \psi^*(z', \boldsymbol{u}) \psi(z, \boldsymbol{u} + [z']) & \text{for} \quad |z| \leqslant |z'| \\ \dfrac{1}{z} \psi^*(z', \boldsymbol{u} - [z]) \psi(z, \boldsymbol{u}) & \text{for} \quad |z'| \leqslant |z|. \end{cases}$$

Proof. Let us first show that

$$\Psi(z, z', \boldsymbol{u}) = \frac{I_N}{z - z'} + \mathcal{O}(1) \qquad z \longrightarrow z'. \tag{8}$$

To do that, we note the following basic relation:

$$\psi_0(z, \boldsymbol{u} + [z']) = \frac{z'}{z' - z}\psi_0(z, \boldsymbol{u}) \qquad |z'| \geqslant |z|$$

where for $|z| = |z'|$ we define

$$\psi_0(z, \boldsymbol{u} + [z']) := \lim_{\epsilon \longrightarrow 0^+} \psi_0(z, \boldsymbol{u} + [(1 + \epsilon)z']).$$

Hence, by setting $\boldsymbol{u} \to \boldsymbol{u} + [z']$ and $\boldsymbol{u}' \to \boldsymbol{u}$ in the bilinear identity (4) and by calculating the residues of the integrand at $z = z'$ and ∞ one gets

$$\operatorname{res}(\psi(z, \boldsymbol{u} + [z']))_{z=z'}\psi^*(z', \boldsymbol{u}) = -z' I_N. \tag{9}$$

If we instead set $\boldsymbol{u}' \to \boldsymbol{u} - [z']$ and $\boldsymbol{u} \in \mathcal{U}(\infty)^N$, we obtain

$$\psi(z', \boldsymbol{u})\operatorname{res}(\psi^*(z, \boldsymbol{u} - [z']))_{z=z'} = -z' I_N. \tag{10}$$

These relations imply at once that

$$-\operatorname{res}\left(\frac{1}{z'}\psi^*(z', \boldsymbol{u})\psi(z, \boldsymbol{u} + [z'])\right)_{z=z'} = \operatorname{res}\left(\frac{1}{z}\psi^*(z', \boldsymbol{u} - [z])\psi(z, \boldsymbol{u})\right)_{z=z'} = I_N.$$

Therefore, (8) follows.

Let us next prove that for $|z| = |z'|$

$$-\frac{1}{z'}\psi^*(z', \boldsymbol{u})\psi(z, \boldsymbol{u} + [z']) = \frac{1}{z}\psi^*(z', \boldsymbol{u} - [z])\psi(z, \boldsymbol{u}). \tag{11}$$

From (9) and (10) we deduce that

$$z\operatorname{res}(\psi^*(z', \boldsymbol{u})\psi(z, \boldsymbol{u} + [z']))_{z=z'} = z'\operatorname{res}(\psi^*(z', \boldsymbol{u} - [z])\psi(z, \boldsymbol{u}))_{z'=z}$$

or, equivalently

$$\psi_0(z', \boldsymbol{u})^{-1}\chi^*(z', \boldsymbol{u})\chi(z', \boldsymbol{u} + [z'])\psi_0(z', \boldsymbol{u}) = \psi_0(z, \boldsymbol{u})^{-1}\chi^*(z, \boldsymbol{u} - [z])\chi(z, \boldsymbol{u})\psi_0(z, \boldsymbol{u}).$$

Hence, in the limit $z \longrightarrow \infty$ we find

$$\chi^*(z', \boldsymbol{u})\chi(z', \boldsymbol{u} + [z']) = I_N. \tag{12}$$

If we set now $\boldsymbol{u} \to \boldsymbol{u} + [z']$ and $\boldsymbol{u}' \to \boldsymbol{u} - [z'']$ in (4), then by calculating the residues of the integrand at $z = z'$ and $z = z''$ we get

$$\chi(z', \boldsymbol{u} + [z'])\chi^*(z', \boldsymbol{u} - [z'']) = \chi(z'', \boldsymbol{u} + [z'])\chi^*(z'', \boldsymbol{u} - [z'']). \tag{13}$$

By using (12) we can rewrite (13) as

$$\chi^*(z', \boldsymbol{u} - [z''])\chi(z'', \boldsymbol{u}) = \chi^*(z', \boldsymbol{u})\chi(z'', \boldsymbol{u} + [z']).$$

This identity immediately leads to (11). The rest of the proof is trivial. □

We now proceed to derive the main property of the Cauchy propagator in connection with the theory of conjugate nets.

Theorem 2. *The entries of the Cauchy propagator satisfy the differential equation*

$$\frac{\partial \Psi_{jk}}{\partial u_i}(z, z', \boldsymbol{u}) = \psi^*_{ji}(z', \boldsymbol{u})\psi_{ik}(z, \boldsymbol{u}). \tag{14}$$

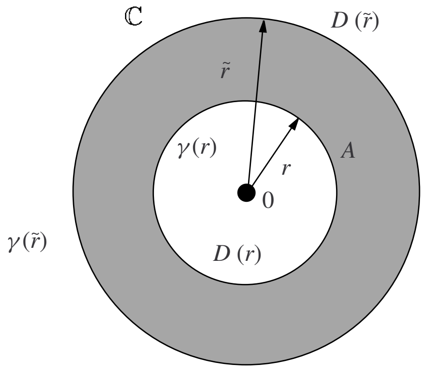

Figure 1. The two discs $D(r)$ and $D(\tilde{r})$.

Proof. For a given $z' \in \mathbb{C}\backslash D(r)$ both members of (14) are analytic functions of z on $\mathbb{C}\backslash D(r)$. Furthermore, as $z \longrightarrow \infty$

$$\frac{\partial \Psi}{\partial u_i}(z, z', \boldsymbol{u}) = \frac{\partial}{\partial u_i}\left[\frac{1}{z}\psi^*(z', \boldsymbol{u} - [z])\psi(z, \boldsymbol{u})\right] = \left[\psi^*(z', \boldsymbol{u})E_i + \mathcal{O}\left(\frac{1}{z}\right)\right]\psi_0(z, \boldsymbol{u}).$$

As $\frac{\partial\Psi}{\partial u_i}\psi_0^{-1}$ is analytic on $\mathbb{C}\backslash D(r)$, its Laurent series expansion near $z = \infty$ can be extended to $\gamma(r)$. Hence

$$\mathfrak{N}\left[\frac{\partial \Psi}{\partial u_i}\psi_0^{-1}\right] = \psi^*(z', \boldsymbol{u})E_i$$

and therefore

$$\frac{\partial \Psi}{\partial u_i}(z, z', \boldsymbol{u}) = \psi^*(z', \boldsymbol{u})E_i\psi(z, \boldsymbol{u})$$

which proves (14). □

As a consequence of (14) the net function of the conjugate net with tangent vectors and Lamé coefficients given by $(\boldsymbol{X}_i)_j := \boldsymbol{X}_{ij}$ and $H_i = \boldsymbol{H}_{li}$, $(l = 1, \ldots, N)$, respectively, is given by the lth row of the matrix function

$$\boldsymbol{x} := \int_{\mathbb{C}\times\mathbb{C}} M(z')\Psi(z, z')N(z)\,\mathrm{d}^2 z\,\mathrm{d}^2 z' + \boldsymbol{x}_0 \tag{15}$$

where $\boldsymbol{x}_0$ is an arbitrary constant matrix.

2.3. Dressing method for conjugate nets

Let $D(r)$ and $D(\tilde{r})$ be two disks with $r < \tilde{r}$ and respective boundaries $\gamma(r)$ and $\gamma(\tilde{r})$. We will denote by A the circular annulus $D(\tilde{r})\backslash D(r)$. See figure 1.

One can define a correspondence between Grassmannians as in the following definition.

Definition 6. *Given a matrix distribution $R = R(z, z')$ with support in $A \times A$, we define an associated dressing transformation*

$$Gr_{\gamma(r)} \longrightarrow Gr_{\gamma(\tilde{r})} \qquad W \mapsto \tilde{W} \tag{16}$$

where for every $W \in Gr_{\gamma(r)}$, the corresponding $\tilde{W} \in Gr_{\gamma(\tilde{r})}$ is defined as the set of boundary values on $\gamma(\tilde{r})$ of matrix functions $w = w(z)$ defined on A such that:

(1) The restriction of w to $\gamma(r)$ is an element of W.

(2) The following $\bar{\partial}$-equation on A is satisfied:

$$\frac{\partial w}{\partial \bar{z}}(z) = \int_A w(z')R(z', z)\,\mathrm{d}^2z' \qquad z \in A.$$

As we are going to show there are wide classes of data $R(z, z')$ for which the dressing transformation (16) is well defined.

In view of the direct relationship between Cauchy propagators and conjugate nets, our main concern now is to learn how they transform under (16). It is proved below that, in the separable case, they do according to the classical geometrical fundamental transformation.

Let Ψ and $\tilde{\Psi}$ be the Cauchy propagators associated with W and $\tilde{W}$, respectively. It is clear that $\tilde{\Psi}$ satisfies

$$\frac{\partial \tilde{\Psi}}{\partial \bar{z}}(z, z') = \pi\delta(z - z') + \int_A \tilde{\Psi}(z'', z')R(z'', z)\,\mathrm{d}^2z'' \qquad |z| > r \qquad |z'| > \tilde{r}. \tag{17}$$

The problem to address here is how to express $\tilde{\Psi}$ in terms of Ψ. To this end we try the ansatz

$$\tilde{\Psi}(z, z') = \Psi(z, z') + \int_A c(z', z'')\Psi(z, z'')\,\mathrm{d}^2z'' \tag{18}$$

for some matrix distribution $c(z, z')$. Introducing (18) in (17) and recalling that in $\mathbb{C}\backslash D(r)$ one has $\bar{\partial}\Psi(z, z') = \pi\delta(z - z')$, we conclude that

$$c(z', z) = \frac{1}{\pi}\int_A \tilde{\Psi}(z'', z')R(z'', z)\,\mathrm{d}^2z''. \tag{19}$$

Now, using (18) in (19) we get

$$c(z', z) = \frac{1}{\pi}\int_A \Psi(z'', z')R(z'', z)\,\mathrm{d}^2z'' + \frac{1}{\pi}\int_{A\times A} c(z', z''')\Psi(z'', z''')R(z'', z)\,\mathrm{d}^2z''\,\mathrm{d}^2z'''. \tag{20}$$

This integral equation for $c(z, z')$ can be solved by standard means when R is a separable kernel

$$R(z, z') = \pi\sum_{k=1}^{m}\sum_{l=1}^{n} f_k(z)C_{k\ell}g_\ell(z') \tag{21}$$

where $C_{k\ell}$ are $N \times N$ constant complex matrices, and f_k, g_ℓ are scalar distributions. To describe the solution of (20) some notation conventions are helpful.

Definition 7. *We shall use the following notation:*

$$\mu_k(z) := \int_A f_k(z')\Psi(z', z)\,\mathrm{d}^2z' \qquad k = 1, \ldots, m$$

$$\nu_\ell(z) := \int_A \Psi(z, z')g_\ell(z')\,\mathrm{d}^2z' \qquad \ell = 1, \ldots, n$$

$$\omega_{\ell k} := \int_{A\times A} f_k(z')\Psi(z', z'')g_\ell(z'')\,\mathrm{d}^2z'\,\mathrm{d}^2z'' \qquad k = 1, \ldots, m \qquad \ell = 1, \ldots, n$$

as well as

$$\boldsymbol{\mu} := (\mu_1, \ldots, \mu_m) : A \to M_{N\times mN}(\mathbb{C}) \qquad \boldsymbol{\nu} := \begin{pmatrix}\nu_1\\ \vdots \\ \nu_n\end{pmatrix} : A \to M_{nN\times N}(\mathbb{C})$$

$$\boldsymbol{C} := (C_{kl}) \in M_{mN\times nN}(\mathbb{C}) \qquad \boldsymbol{\omega} = (\omega_{\ell k}) \in M_{nN\times mN}(\mathbb{C}).$$

We also denote

$$\lambda_k(z) := \int_A c(z, z')\mu_k(z')\,\mathrm{d}^2z' \qquad k = 1, \ldots, m$$
$$\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_m) : A \to M_{N\times mN}(\mathbb{C})$$
$$\boldsymbol{g} := \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} : A \to M_{nN\times N}(\mathbb{C}).$$

With these expressions at hand we prove the following theorem.

Theorem 3. *In the separable case given by (21) the dressed Cauchy kernel reads*

$$\tilde{\Psi}(z, z') = \Psi(z, z') + \boldsymbol{\mu}(z')\boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}\boldsymbol{\nu}(z). \tag{22}$$

Proof. Introducing (21) in (20) we find

$$c(z', z) = \boldsymbol{\mu}(z')\boldsymbol{C}\boldsymbol{g}(z) + \int_A c(z', z''')\boldsymbol{\mu}(z''')\boldsymbol{C}\boldsymbol{g}(z)\,\mathrm{d}^2z''' \tag{23}$$

or in terms of λ

$$c(z', z) = (\boldsymbol{\mu}(z') + \boldsymbol{\lambda}(z'))\boldsymbol{C}\boldsymbol{g}(z). \tag{24}$$

Hence, if we insert (24) in (23) and assume that all the entries in $\boldsymbol{g}$ are functionally independent, we get

$$\boldsymbol{\lambda}(z')\boldsymbol{C} = \boldsymbol{\mu}(z')\boldsymbol{C}\boldsymbol{\omega}\boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}$$

and therefore, using (24), we derive

$$\begin{aligned} c(z', z) &= \boldsymbol{\mu}(z')\boldsymbol{C}(I_{nN} + \boldsymbol{\omega C}(I_{nN} - \boldsymbol{\omega C})^{-1})\boldsymbol{g}(z) \\ &= \boldsymbol{\mu}(z')\boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}\boldsymbol{g}(z). \end{aligned}$$

By inserting this expression in (18) we deduce (22). □

There are several observations in order:

(1) A similar analysis can be applied to derive the formulae for the dressed wave $\psi(z)$ function, adjoint wavefunction $\psi^*(z)$ and the matrix of rotation coefficients β. They turn out to be

$$\tilde{\psi}(z) = \psi(z) + \boldsymbol{\varphi}\boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}\boldsymbol{\nu}(z)$$
$$\tilde{\psi}^*(z) = \psi^*(z) + \boldsymbol{\mu}(z)\boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}\boldsymbol{\varphi}^*$$
$$\tilde{\beta} = \beta + \boldsymbol{\varphi}\boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}\boldsymbol{\varphi}^*$$

with

$$\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_m) \qquad \varphi_k = \int_A f_k(z)\psi(z)\,\mathrm{d}^2z$$
$$\boldsymbol{\varphi}^* := \begin{pmatrix} \varphi_1^* \\ \vdots \\ \varphi_n^* \end{pmatrix} \qquad \varphi_\ell^* = \int_A \psi^*(z)g_\ell(z)\,\mathrm{d}^2z.$$

(2) The dressing of the parallel nets $\boldsymbol{x}$ of (15) is

$$\tilde{\boldsymbol{x}} := \boldsymbol{x} + \boldsymbol{M}\boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}\boldsymbol{N}$$

where

$$\boldsymbol{M} = \int_{\mathbb{C}} M(z)\boldsymbol{\mu}(z)\,\mathrm{d}^2z \qquad \boldsymbol{N} = \int_{\mathbb{C}} \boldsymbol{\nu}(z)N(z)\,\mathrm{d}^2z.$$

The dressed tangent vectors and Lamé coefficients are obtained from

$$\tilde{\boldsymbol{X}} = \boldsymbol{X} + \varphi \boldsymbol{C}(I_{nN} - \boldsymbol{\omega C})^{-1}\boldsymbol{N}$$
$$\tilde{\boldsymbol{H}} = \boldsymbol{H} + \boldsymbol{MC}(I_{nN} - \boldsymbol{\omega C})^{-1}\varphi^*.$$

(3) All these formulae are just a generalization of the vectorial fundamental transformation [7, 11, 12, 15, 17], as it is deduced from the following description of transformed and transformation data. We have a row of $\boldsymbol{x}$ say the lth $\boldsymbol{x}^{(l)}$, representing a conjugate net with tangent vectors given by the rows of $\boldsymbol{X}$, say $\boldsymbol{X}_j$, $j = 1, \ldots, N$, and Lamé coefficients given by $H_j := \boldsymbol{H}_{lj}$; the points of the net $\boldsymbol{x}^{(l)}$ satisfy $\partial \boldsymbol{x}^{(l)}/\partial u_j = \boldsymbol{X}_j H_j$. The transformation data are the N rows of $\varphi \boldsymbol{C}$: $\{\boldsymbol{\varphi}_j \boldsymbol{C}\}_{j=1}^N$ and the N columns of $\boldsymbol{\varphi}^*$: $\{\boldsymbol{\varphi}_j^*\}_{j=1}^N$. Here $I_{nN} - \boldsymbol{\omega C}$ is a potential for these transformation data: $\partial/\partial u_j(I_{nN} - \boldsymbol{\omega C}) = \boldsymbol{\varphi}_j^* \boldsymbol{\varphi}_j \boldsymbol{C}$. Moreover, $\boldsymbol{M}_l \boldsymbol{C}$ and $\boldsymbol{N}$ satisfy the corresponding potential equations:

$$\frac{\partial(\boldsymbol{M}_l\boldsymbol{C})}{\partial u_j} = H_j \boldsymbol{\varphi}_j \boldsymbol{C} \qquad \frac{\partial \boldsymbol{N}}{\partial u_j} = \boldsymbol{\varphi}_j^* \boldsymbol{X}_j.$$

The generalization provided by the dressing transformations involves one new ingredient: the transformation data $\boldsymbol{\varphi}_j$ and $\boldsymbol{\varphi}_j^*$ are vectors in linear spaces with different dimensions.

3. Explicit examples of conjugate nets

This section is devoted to show the effectiveness of the dressing method we have presented. First, we consider how the basic Cartesian net appears in relation with ψ_0, i.e. with the zero background of the KP hierarchy. This Cartesian net will be the departure point for our dressing method: we show how the Cartesian net transforms under a general dressing transformation, then we consider the case $n = m = 1$, the simplest separable kernel. In the latter case we provide bounds for ensuring that the new coordinates are non-singular and constitute a locally regular set of conjugate coordinates.

Despite the fact that two-dimensional nets in the plane are trivial examples of conjugate nets, their analysis is interesting because it captures the essence of the behaviour in higher dimensions. Perhaps the most interesting case is the three-dimensional one, where the problem of constructing triply conjugate systems of surfaces is far from being trivial.

Among the wide families of examples of explicit conjugate nets, provided by the simplest separable case of the dressing method, we consider two particular ones:

- Periodic conjugate nets: here the spectral input data are real combinations of Dirac delta functions with point support over the imaginary axis and the net is periodic and of trigonometric type. These examples can be extended, as we will show in the next paper of this series, to functions of elliptic type or even with a more general periodic behaviour.
- Hermite conjugate nets: the spectral distributions are now of Gaussian type. In this case the nets are expressed in terms of Gaussian, error and Hermite functions. In all cases, the coordinate lines (in two dimensions) or surfaces (in three dimensions) are asymptotically Cartesian. However, the most relevant aspect of a very large subset in the Hermite family of conjugate nets, is that these nets are deformed Cartesian nets with a Gaussian perturbation localized at the origin; i.e., in the two-dimensional case for any $\epsilon > 0$ there exists a disc $D(0, \delta)$ centred at the origin of radius δ, such that for any pair of undressed coordinate line $\mathfrak{c}$ and its corresponding dressed coordinate curve $\tilde{\mathfrak{c}}$, both nonintersecting $D(0, \delta)$, the maximum length of the orthogonal segments to $\mathfrak{c}$, with ends lying on $\mathfrak{c}$ and $\tilde{\mathfrak{c}}$ is less than ϵ. In three dimensions a similar statement for coordinate surfaces and planes holds. Again, this behaviour is not exclusive to these Hermite nets, and there are other families with a

similar localization, for example, instead of Gaussian type of rational type that we shall present elsewhere.

We should emphasize that all these examples can be ensured to be regular and singularity free, provided the relevant parameters are small enough.

3.1. Dressing a Cartesian net

The points of a Cartesian net are of the form by $\sum_{i=1}^{N} \boldsymbol{v}_i u_i + \boldsymbol{v}$, where $\boldsymbol{v}_i$, $\boldsymbol{v}$ are constant vectors, with $\{\boldsymbol{v}_i\}_{i=1}^{N}$ a linear basis in $\mathbb{R}^N$, the tangent vectors can be taken as $\boldsymbol{X}_i = \boldsymbol{v}_i/H_i$ while the Lamé coefficients are arbitrary constants $H_1, \ldots, H_N$ (obviously $\beta_{ij} = 0$). A Cartesian net can be recovered from the element $H^+_{\gamma(r)}$ of $\mathrm{Gr}_{\gamma(r)}$. In this case, one has

$$\psi(z) = \psi_0(z) \qquad \psi^*(z) = \psi_0(z)^{-1} \qquad \Psi(z, z') = \frac{1}{z - z'} \psi_0(z')^{-1} \psi_0(z).$$

Thus, a set of parallel conjugate nets is given by the rows of

$$\boldsymbol{x} = \int_{\mathbb{C}^2} M(z') \frac{\psi_0(z')^{-1}\psi_0(z)}{z - z'} N(z) \, \mathrm{d}^2 z \, \mathrm{d}^2 z' + \boldsymbol{x}_0.$$

In particular, for $M(z) = \delta(z - p)A$, $N(z) = \delta(z - q)B$, with $A, B \in M_N(\mathbb{C})$ and $\boldsymbol{x}_0 = AB/(p - q) + \boldsymbol{x}'_0$, we have

$$\boldsymbol{x} = A\psi_0(p)^{-1} \frac{\psi_0(q) - \psi_0(p)}{q - p} B + \boldsymbol{x}'_0.$$

Now, by setting $p, q \to 0$, it becomes

$$\boldsymbol{x}^{(l)}(\boldsymbol{u}) = \boldsymbol{A}_l \psi_0(0, \boldsymbol{u})^{-1} \frac{\partial \psi_0}{\partial z}(0, \boldsymbol{u}) B + \boldsymbol{x}'_0 = \sum_{j=1}^{N} A_{lj} u_j \boldsymbol{B}_j + \boldsymbol{x}'^{(l)}_0$$

which represents a Cartesian net with $H_j = A_{lj}$ and $\boldsymbol{X}_j = \boldsymbol{B}_j$, $j = 1, \ldots, N$. The potentials $\boldsymbol{M}$ and $\boldsymbol{N}$ are $A\boldsymbol{\mu}(0)$ and $\boldsymbol{\nu}(0)B$, respectively.

Proposition 1. *The dressing of a Cartesian net gives a new conjugate net defined by*

$$\begin{aligned}
\tilde{\boldsymbol{x}}^{(l)} &:= \boldsymbol{x}^{(l)} + \boldsymbol{A}_l \boldsymbol{\mu}(0) \boldsymbol{C}(I_{nN} - \boldsymbol{\omega}\boldsymbol{C})^{-1} \boldsymbol{\nu}(0) B \\
\tilde{\boldsymbol{X}}_j &:= \boldsymbol{B}_j + \boldsymbol{\varphi}_j \boldsymbol{C}(I_{nN} - \boldsymbol{\omega}\boldsymbol{C})^{-1} \boldsymbol{\nu}(0) B \\
\tilde{H}_j &:= A_{lj} + \boldsymbol{A}_l \boldsymbol{\mu}(0) \boldsymbol{C}(I_{nN} - \boldsymbol{\omega}\boldsymbol{C})^{-1} \boldsymbol{\varphi}^*_j \\
\tilde{\beta}_{jk} &= \boldsymbol{\varphi}_j \boldsymbol{C}(I_{nN} - \boldsymbol{\omega}\boldsymbol{C})^{-1} \boldsymbol{\varphi}^*_k.
\end{aligned}$$

The standard Cartesian net corresponds to the choice $A_{l1} = \ldots = A_{lN} = 1$ and $B = I_N$. For $n = m = 1$, the spectral data are the distributions $f(z)$ and $g(z)$. If we define

$$\begin{aligned}
\phi_i(u_i) &:= \int_A \mathrm{e}^{z u_i} f(z) \, \mathrm{d}^2 z \qquad & \phi^*_i(u_i) &:= \int_A \mathrm{e}^{-z u_i} g(z) \, \mathrm{d}^2 z \\
\mu_i(u_i) &:= \int_A \frac{\mathrm{e}^{z u_i}}{z} f(z) \, \mathrm{d}^2 z \qquad & \nu_i(u_i) &:= -\int_A \frac{\mathrm{e}^{-z u_i}}{z} g(z) \, \mathrm{d}^2 z \\
\omega_i(u_i) &:= \int_{A \times A} \frac{\mathrm{e}^{(z - z')u_i}}{z - z'} f(z) g(z') \, \mathrm{d}^2 z \, \mathrm{d}^2 z'
\end{aligned}$$

and write $C = (c_{ij})$, then the dressing transformation gives us the following proposition.

Proposition 2. *The next data characterize a conjugate net*

$$(\tilde{x}_1, \dots, \tilde{x}_N) = (u_1, \dots, u_N) + (\mu_1, \dots, \mu_N) C (I_N - \mathrm{diag}(\omega_1, \dots, \omega_N) C)^{-1} \mathrm{diag}(\nu_1, \dots, \nu_N)$$

$$\tilde{\boldsymbol{X}}_j := \boldsymbol{e}_j [I_N + \mathrm{diag}(\phi_1, \dots, \phi_N) C (I_N - \mathrm{diag}(\omega_1, \dots, \omega_N) C)^{-1} \mathrm{diag}(\nu_1, \dots, \nu_N)]$$

$$\tilde{H}_j := 1 + (\mu_1, \dots, \mu_N) C (I_N - \mathrm{diag}(\omega_1, \dots, \omega_N) C)^{-1} \mathrm{e}_j^{\mathrm{t}} \phi_j^*$$

$$\tilde{\beta}_{jk} = \phi_j \phi_k^* \boldsymbol{e}_j C (I_N - \mathrm{diag}(\omega_1, \dots, \omega_N) C)^{-1} \boldsymbol{e}_k^{\mathrm{t}}$$

where $\{\boldsymbol{e}_j\}_{j=1}^N$ *stands for the canonical basis in* $\mathbb{R}^N$.

An important issue regarding this new conjugate net is the local regularity of the change of variables; i.e. if

$$\frac{\partial(\tilde{x}_1, \dots, \tilde{x}_N)}{\partial(u_1, \dots, u_N)} \neq 0.$$

By recalling that $\frac{\partial \tilde{\boldsymbol{x}}}{\partial u_j} = \tilde{H}_j \tilde{\boldsymbol{X}}_j$, it is clear that the Jacobian is

$$\frac{\partial(\tilde{x}_1, \dots, \tilde{x}_N)}{\partial(u_1, \dots, u_N)} = \det(\tilde{\boldsymbol{X}}_1, \dots, \tilde{\boldsymbol{X}}_N) \prod_{j=1}^N H_j.$$

Thus, the transformation is not locally regular when either $\det(\tilde{\boldsymbol{X}}_1, \dots, \tilde{\boldsymbol{X}}_N)$ or any of the Lamé coefficients $\tilde{H}_j$ vanishes. On the one hand

$$\det(\tilde{\boldsymbol{X}}_1, \dots, \tilde{\boldsymbol{X}}_N) = \det(I_N + \phi \Omega \nu)$$

with $\phi := \mathrm{diag}(\phi_1, \dots, \phi_N)$, $\Omega := C(I_N - \mathrm{diag}(\omega_1, \dots, \omega_N) C)^{-1}$ and $\nu := \mathrm{diag}(\nu_1, \dots, \nu_N)$). This means that only when -1 is an eigenvalue of $\phi \Omega \nu$ the determinant vanishes. We can prevent this happening in many ways, for example, the Hirsch bound told us that if λ is an eigenvalue then $|\lambda| \leqslant N \max_{i,j}(|\phi_i \Omega_{ij} \nu_j|)$; hence, we may conclude that if $\forall i, j = 1, \dots, N$ $|\phi_i||\Omega_{ij}||\nu_j| < 1/N$ then

$$\det(\tilde{\boldsymbol{X}}_1, \dots, \tilde{\boldsymbol{X}}_N) \neq 0.$$

On the other hand as $\tilde{H}_j = 1 + \sum_{i=1}^N \mu_i \Omega_{ij} \phi_j^*$ it is obvious that if$|\mu_i||\Omega_{ij}||\phi_j^*| < 1/N$, $\forall i, j = 1, \dots, N$, then $\prod_{l=1}^N \tilde{H}_l \neq 0$.

Another important aspect to be considered is the possible presence of singularities in the net, which appear only if $(I_N - \omega C)$ is not invertible; i.e. when

$$\Delta := \det(I_N - \omega C) = 1 + \sum_{k=1}^N (-1)^k \sum_{i_1, \dots, i_k} \det(C_{i_1 \dots i_k}) \omega_{i_1} \dots \omega_{i_k}$$

has a zero. Here $i_1, \dots i_k$ are different numbers taken form $1, \dots, N$ and $C_{i_1 \dots i_k}$ is the matrix built up from the $i_1, \dots, i_k$ rows and columns of C. The same Hirsch argument applies an we conclude that if $|\omega_i||c_{ij}| \leqslant 1/N$, $\forall i, j = 1, \dots, N$, then $\Delta \neq 0$.

3.2. Examples in two and three dimensions

For the simplest $(n = m = 1)$ bidimensional case $N = 2$ we have

$$\Delta(u_1, u_2) = 1 - c_{11}\omega_1(u_1) - c_{22}\omega_2(u_2) + |C|\omega_1(u_1)\omega_2(u_2).$$

The dressed net, renormalized tangent vectors, Lamé and rotation coefficients are given by

$$\begin{aligned}\tilde{x}_i &= u_i + \frac{1}{\Delta}[\mu_i(c_{ii} - |C|\omega_j) + \mu_j c_{ji}]\nu_i \\ \tilde{\boldsymbol{X}}_i &= \left(1 + \frac{c_{ii} - |C|\omega_j}{\Delta}\phi_i\nu_i\right)\boldsymbol{e}_i + \frac{c_{ij}}{\Delta}\phi_i\nu_j\boldsymbol{e}_j \\ \tilde{H}_i &= 1 + \frac{1}{\Delta}[\mu_i(c_{ii} - |C|\omega_j) + \mu_j c_{ji}]\phi_i^* \\ \tilde{\beta}_{ij} &= \frac{c_{ij}}{\Delta}\phi_i\phi_j^*\end{aligned}$$

with i, j cyclic. Here the matrix Ω is defined by

$$\Omega_{ii} = \frac{c_{ii} - |C|\omega_j}{\Delta} \qquad \Omega_{ij} = \frac{c_{ij}}{\Delta} \qquad i \neq j.$$

In order to consider the three-dimensional case $N = 3$, the cofactors κ_{ij} of the coefficient c_{ij} are required. We have

$$\Delta = 1 - c_{11}\omega_1 - c_{22}\omega_2 - c_{33}\omega_3 + \kappa_{33}\omega_1\omega_2 + \kappa_{22}\omega_1\omega_3 + \kappa_{11}\omega_2\omega_3 - |C|\omega_1\omega_2\omega_3$$

and the corresponding formulae are

$$\begin{aligned}\tilde{x}_i &= u_i + \frac{1}{\Delta}\left[\mu_i(c_{ii} - \kappa_{jj}\omega_k - \kappa_{kk}\omega_j + |C|\omega_j\omega_k) + \sum_{j\neq i}\mu_j(c_{ji} + \kappa_{ij}\omega_k)\right]\nu_i \\ \tilde{\boldsymbol{X}}_i &= \left(1 + \frac{c_{ii} - \kappa_{jj}\omega_k - \kappa_{kk}\omega_j + |C|\omega_j\omega_k}{\Delta}\phi_i\nu_i\right)\boldsymbol{e}_i + \sum_{j\neq i}\frac{c_{ij} + \kappa_{ji}\omega_k}{\Delta}\phi_i\nu_j\boldsymbol{e}_j \\ \tilde{H}_i &= 1 + \frac{1}{\Delta}\left[\mu_i(c_{ii} - \kappa_{jj}\omega_k - \kappa_{kk}\omega_j + |C|\omega_j\omega_k) + \sum_{j\neq i}\mu_j(c_{ji} + \kappa_{ij}\omega_k)\right]\phi_i^* \\ \tilde{\beta}_{ij} &= \frac{c_{ij} + \kappa_{ji}\omega_k}{\Delta}\phi_i\phi_j^*\end{aligned}$$

with i, j, k cyclic. Here, Ω is

$$\Omega_{ii} = \frac{c_{ii} - \kappa_{jj}\omega_k - \kappa_{kk}\omega_j + |C|\omega_j\omega_k}{\Delta} \qquad \Omega_{ij} = \frac{c_{ij} + \kappa_{ji}\omega_k}{\Delta} \qquad i \neq j.$$

3.2.1. Periodic conjugate nets of trigonometric type. An example of periodic net is provided by the choice

$$\begin{aligned}f(z) &= \frac{A}{2}(\mathrm{e}^{\mathrm{i}\alpha}\delta(z - \mathrm{i}p) + \mathrm{e}^{-\mathrm{i}\alpha}\delta(z + \mathrm{i}p)) \\ g(z) &= \frac{B}{2}(\mathrm{e}^{\mathrm{i}\beta}\delta(z - \mathrm{i}q) + \mathrm{e}^{-\mathrm{i}\beta}\delta(z + \mathrm{i}q))\end{aligned}$$

with $p, q, \alpha, \beta \in \mathbb{R}$, so that

$$\begin{aligned}\phi_j(u_j) &= A\cos(pu_j + \alpha) \qquad \phi_j^*(u_j) = B\cos(qu_j - \beta) \\ \mu_j(u_j) &= \frac{A}{p}\sin(pu_j + \alpha) \qquad \nu_j(u_j) = \frac{B}{q}\sin(qu_j - \beta) \\ \omega_j(u_j) &= \frac{AB}{2}\left(\frac{\sin((p-q)u_j + \alpha + \beta)}{p - q} + \frac{\sin((p+q)u_j + \alpha - \beta)}{p + q}\right).\end{aligned}$$

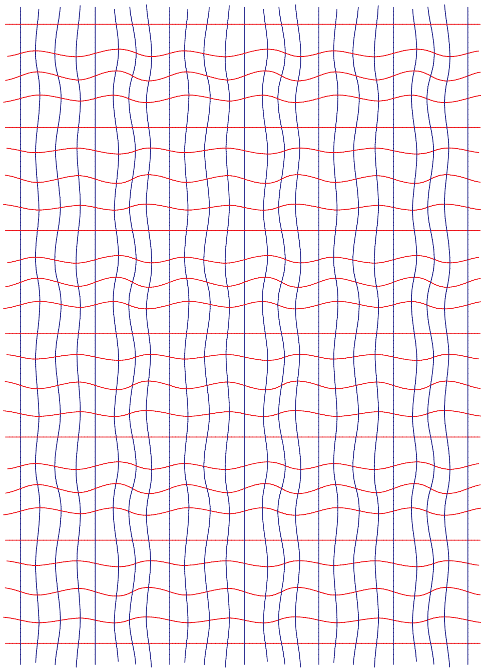

Figure 2. Plot of the coordinate lines for $N = 2$ and $C = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $p = 2, q = 1, \alpha = \beta = 0$, $A = B = 1$.

For $N = 2$ and $C = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $p = 2$, $q = 1$, $\alpha = \beta = 0$, $A = B = 1$, the net is given by

$$\begin{aligned} x_1(u_1, u_2) := u_1 &+ 3(-12\sin(2u_2)\sin(u_1) + (12 - 3\sin(u_2) \\ &- \sin(3u_2))\sin(2u_1)\sin(u_1))\{288 - 12(3\sin(u_1) + \sin(3u_1) + 3\sin(u_2) \\ &+ \sin(3u_2)) + (3\sin(u_1) + \sin(3u_1))(3\sin(u_2) + \sin(3u_2))\}^{-1} \\ x_2(u_1, u_2) := u2 &+ 3(12\sin(2u_1)\sin(u_2) + (12 - 3\sin(u_1) \\ &- \sin(3u_1))\sin(2u_2)\sin(u_2))\{288 - 12(3\sin(u_1) + \sin(3u_1) + 3\sin(u_2) \\ &+ \sin(3u_2)) + (3\sin(u_1) + \sin(3u_1))(3\sin(u_2) + \sin(3u_2))\}^{-1} \end{aligned}$$

for which the coordinate lines are plotted in figure 2.

For $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $p = 2$, $q = 1$, $\alpha = \beta = 0$, $A = B = 1$, the net is given by

$$\begin{aligned} x_1(u_1, u_2, u_3) := u_1& \\ &+3((144\sin(2u_3)\sin(u_1) + (12\sin(2u_2) + \sin(2u_1)(3\sin(u_2) \\ &+ \sin(3u_2)))(3\sin(u_3) + \sin(3u_3))\sin(u_1))\{1728 - (3\sin(u_1) \\ &+ \sin(3u_1))(3\sin(u_2) + \sin(3u_2))(3\sin(u_3) + \sin(3u_3))\}^{-1}) \\ x_2(u_1, u_2, u_3) := u_2& \\ &+3((144\sin(2u_1)\sin(u_2) + (12\sin(2u_3) + \sin(2u_2)(3\sin(u_3) \end{aligned}$$

$$
\begin{aligned}
&+\sin(3u_3)))(3\sin(u_1)+\sin(3u_1))\sin(u_2))\{1728-(3\sin(u_1)\\
&+\sin(3u_1))(3\sin(u_2)+\sin(3u_2))(3\sin(u_3)+\sin(3u_3))\}^{-1})\\
x_3(u_1,u_2,u_3) &:= u_3\\
&+3((144\sin(2u_2)\sin(u_3)+(12\sin(2u_1)+\sin(2u_3)(3\sin(u_1)\\
&+\sin(3u_1)))(3\sin(u_2)+\sin(3u_2))\sin(u_3))\{1728-(3\sin(u_1)\\
&+\sin(3u_1))(3\sin(u_2)+\sin(3u_2))(3\sin(u_3)+\sin(3u_3))\}^{-1})
\end{aligned}
$$

and the triply conjugate surfaces u_i = constant, $i = 1, 2, 3$, are plotted in figure 3.

A plot of a generic surface and its conjugate net of coordinate lines is shown in figure 4.

We notice that the trigonometric family has the straightforward extension

$$
\begin{aligned}
f(z) &= \sum_{k=1}^{r}\frac{A_k}{2}(\mathrm{e}^{\mathrm{i}\alpha_k}\delta(z-\mathrm{i}p_k)+\mathrm{e}^{-\mathrm{i}\alpha_k}\delta(z+\mathrm{i}p_k))\\
g(z) &= \sum_{l=1}^{s}\frac{B_l}{2}(\mathrm{e}^{\mathrm{i}\beta_l}\delta(z-\mathrm{i}q_l)+\mathrm{e}^{-\mathrm{i}\beta_l}\delta(z+\mathrm{i}q_l)).
\end{aligned}
$$

3.2.2. Hermite conjugate nets. The form of the functions ϕ, ϕ^* and μ, ν strongly suggests the use of integral transforms of Fourier and Laplace type. The particular case we are going to analyse here is that giving Hermite functions. The corresponding spectral distributions

$$
\begin{aligned}
f(z) &= Ak/\sqrt{2\pi}\,\delta\left(\frac{z+\bar{z}}{2}\right)z^r\mathrm{e}^{k^2z^2/2}\\
g(z) &= Bl/\sqrt{2\pi}\,\delta\left(\frac{z+\bar{z}}{2}\right)z^s\mathrm{e}^{l^2z^2/2}
\end{aligned}
$$

are concentrated on the imaginary axis. Here A, B, k, l real and r, s non-negative integers. In what follows this choice will be denoted by (r, s).

This example involves the Gaussian distribution and functions related to it: the error function $\operatorname{erf}(u) := 2/\sqrt{\pi}\int_0^u \mathrm{e}^{-t^2}\,\mathrm{d}t$ and the Hermite polynomials $H_n(u)$ defined by

$$
H_n(u) = \mathrm{e}^{u^2}\frac{\mathrm{d}^n\mathrm{e}^{-u^2}}{\mathrm{d}u^n} = \sum_{k=0}^{[n/2]}(-1)^k\frac{n!}{k!(n-2k)!}(2u)^{n-2k}.
$$

Two useful formulae are

$$
\begin{aligned}
\frac{k}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathrm{i}t)^{-1}\mathrm{e}^{-\frac{k^2t^2}{2}}\mathrm{e}^{\mathrm{i}tu}\mathrm{d}t &= k\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{u}{\sqrt{2}k}\right)\\
\frac{k}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathrm{i}t)^n\mathrm{e}^{-\frac{k^2t^2}{2}}\mathrm{e}^{\mathrm{i}tu}\,\mathrm{d}t &= \frac{1}{(\sqrt{2}k)^n}H_n\left(-\frac{u}{\sqrt{2}k}\right)\mathrm{e}^{-\frac{u^2}{2k^2}} \qquad n \geqslant 0.
\end{aligned}
$$

After some computations one can derive the following form of the transformation data:

$$
\begin{aligned}
\phi(u) &= \frac{A}{(\sqrt{2}k)^r}H_r\left(-\frac{u}{\sqrt{2}k}\right)\mathrm{e}^{-\frac{u^2}{2k^2}}\\
\phi^*(u) &= \frac{B}{(\sqrt{2}l)^s}H_s\left(\frac{u}{\sqrt{2}l}\right)\mathrm{e}^{-\frac{u^2}{2k^2}}\\
\mu(u) &= \begin{cases} Ak\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{u}{\sqrt{2}k}\right) & r=0\\ \frac{A}{(\sqrt{2}k)^{r-1}}H_{r-1}\left(-\frac{u}{\sqrt{2}k}\right)\mathrm{e}^{-\frac{u^2}{2k^2}} & r>0\end{cases}
\end{aligned}
$$

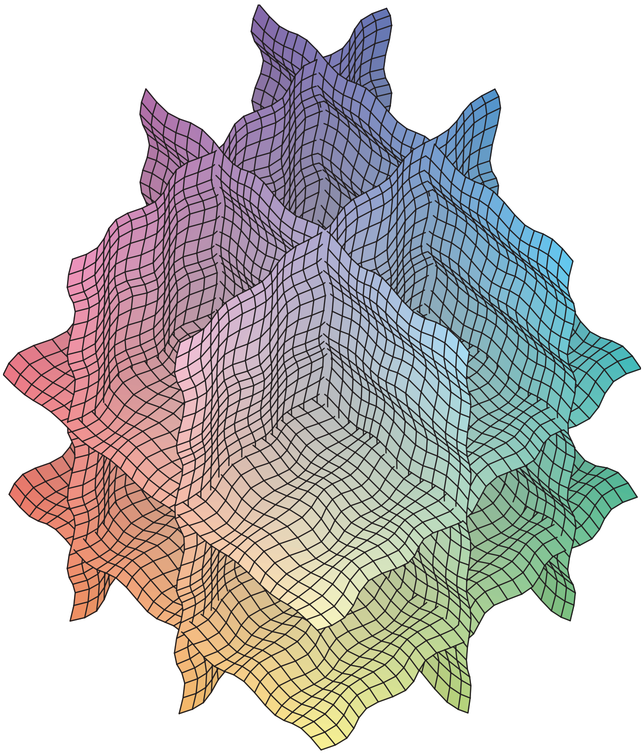

Figure 3. Plot of the triply conjugate surfaces u_i = constant, $i = 1, 2, 3$, for $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $p = 2, q = 1, \alpha = \beta = 0, A = B = 1$.

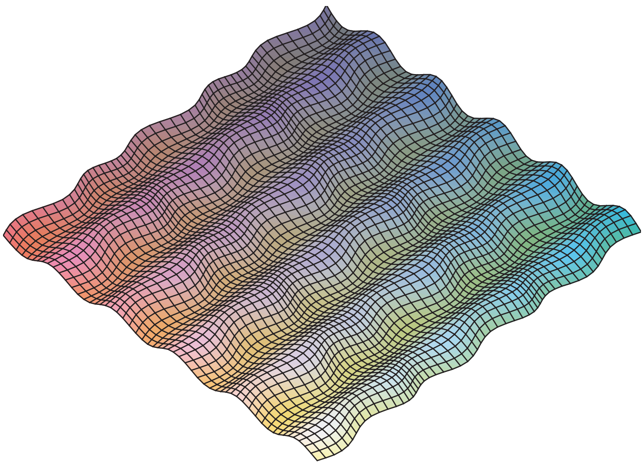

Figure 4. Plot of a generic surface and its conjugate net of coordinate lines $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $p = 2, q = 1, \alpha = \beta = 0, A = B = 1$.

$$\nu(u) = \begin{cases} Bl\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{u}{\sqrt{2}l}\right) & s = 0 \\ -\frac{B}{(\sqrt{2}l)^{s-1}}H_{s-1}\left(\frac{u}{\sqrt{2}l}\right)\mathrm{e}^{-\frac{u^2}{2l^2}} & s > 0 \end{cases}$$

and for $r+s$ even

$$\omega(u) = \frac{1}{2^{r+s}}\frac{(-1)^{\frac{r+s}{2}}kl}{\sqrt{\frac{k^2+l^2}{2}}^{r+s+1}}AB\left[\frac{\sqrt{\pi}}{2}\frac{(r+s)!}{\frac{r+s}{2}!}\operatorname{erf}\left(u\frac{\sqrt{k^2+l^2}}{\sqrt{2}kl}\right)\right.$$
$$-\sum_{p=1}^{\frac{r+s}{2}}\sum_{m+n=2p}(-1)^{n+p}\binom{r}{m}\binom{s}{n}k^{2n-2p}l^{2m-2p}\frac{(r+s-2p)!}{\frac{r+s-2p}{2}!}$$
$$\left.\times H_{2p-1}\left(u\frac{\sqrt{k^2+l^2}}{\sqrt{2}kl}\right)\mathrm{e}^{-u^2\frac{k^2+l^2}{2k^2l^2}}\right]$$

while for $r+s$ odd

$$\omega(u) = \frac{1}{2^{r+s}}\frac{(-1)^{\frac{r+s+1}{2}}}{\sqrt{\frac{k^2+l^2}{2}}^{r+s+1}}AB\sum_{p=0}^{\frac{r+s-1}{2}}\sum_{m+n=2p+1}(-1)^{n+p}\binom{r}{m}\binom{s}{n}$$
$$\times k^{2n-2p}l^{2m-2p}\frac{(r+s-1-2p)!}{\frac{r+s-1-2p}{2}!}H_{2p}\left(u\frac{\sqrt{k^2+l^2}}{\sqrt{2}kl}\right)\mathrm{e}^{-u^2\frac{k^2+l^2}{2k^2l^2}}.$$

An important observation regarding the family of Hermite conjugate nets of type (r, s), with $s > 0$, is that asymptotically the surfaces $u_j = c \longrightarrow \infty$ correspond to $\tilde{x}_j = c$. Hence, asymptotically these nets become the Cartesian net, while there is a Gaussian localized deformation in a neighbourhood of the origin. This follows from the form of $\tilde{x}_j = u_j + \sum_{k=1}^N \mu_k(u_k)\Omega_{kj}(u_1, \ldots, u_N)\nu_j(u_j)$ and the fact that μ_k and Ω_{kj} are bounded functions. Hence for u_j large, the Gaussian decay of ν implies the statement.

We are going to analyse the cases (0, 0) and (0, 3); the second one describes a Hermite net which is asymptotically Cartesian.

- **Case (0, 0): the error net.** We take

$$\mu_j(u_j) = \nu_j(u_j) = \operatorname{erf}\left(\frac{u_j}{\sqrt{2}}\right)$$
$$\omega_j(u_j) = \frac{1}{\sqrt{\pi}}\operatorname{erf}(u_j).$$

For the bidimensional case $N = 2$ with $C = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, the net is given by

$$x_1(u_1, u_2) := u_1 + \sqrt{\pi}\operatorname{erf}\left(\frac{u_1}{\sqrt{2}}\right)\frac{-2\sqrt{\pi}\operatorname{erf}(\frac{u_2}{\sqrt{2}}) + (2\sqrt{\pi} - \operatorname{erf}(u_2))\operatorname{erf}(\frac{u_1}{\sqrt{2}})}{8\pi - 2\sqrt{\pi}\operatorname{erf}(u_1) - 2\sqrt{\pi}\operatorname{erf}(u_2) + \operatorname{erf}(u_1)\operatorname{erf}(u_2)}$$
$$x_2(u_1, u_2) := u_2 + \sqrt{\pi}\operatorname{erf}\left(\frac{u_2}{\sqrt{2}}\right)\frac{2\sqrt{\pi}\operatorname{erf}(\frac{u_1}{\sqrt{2}}) + (2\sqrt{\pi} - \operatorname{erf}(u_1))\operatorname{erf}(\frac{u_2}{\sqrt{2}})}{8\pi - 2\sqrt{\pi}\operatorname{erf}(u_1) - 2\sqrt{\pi}\operatorname{erf}(u_2) + \operatorname{erf}(u_1)\operatorname{erf}(u_2)}$$

and exhibits the plot shown in figure 5.

Notice that asymptotically all the coordinate lines are straight; however, the perturbation of the Cartesian net is not localized. Instead, the bend of the coordinate lines does not decay asymptotically.

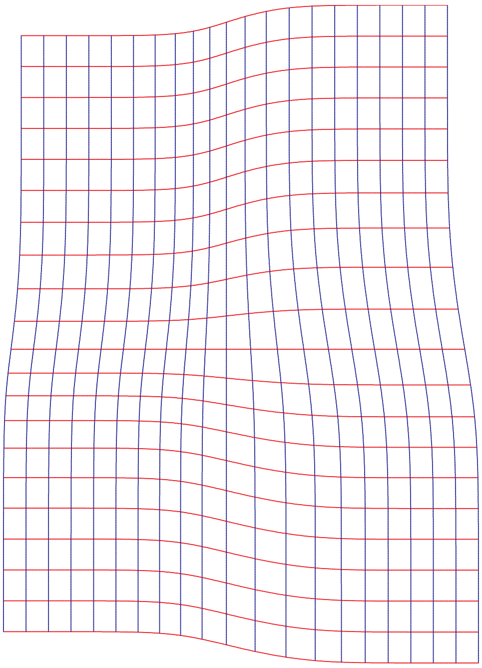

Figure 5. Plot of the net for the bidimensional case $N = 2$ with $C = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

For $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$x_1(u_1, u_2, u_3) = u_1 + \sqrt{\pi}\,\mathrm{erf}\left(\frac{u_1}{\sqrt{2}}\right) \times \frac{4\pi\,\mathrm{erf}(\frac{u_3}{\sqrt{2}}) + 2\mathrm{erf}(\frac{u_2}{\sqrt{2}})\mathrm{erf}(u_3) + \mathrm{erf}(\frac{u_1}{\sqrt{2}})\mathrm{erf}(u_2)\mathrm{erf}(u_3)}{8\sqrt{\pi^3} - \mathrm{erf}(u_1)\mathrm{erf}(u_2)\mathrm{erf}(u_3)}$$

$$x_2(u_1, u_2, u_3) = u_2 + \sqrt{\pi}\,\mathrm{erf}\left(\frac{u_2}{\sqrt{2}}\right) \times \frac{4\pi\,\mathrm{erf}(\frac{u_1}{\sqrt{2}}) + 2\mathrm{erf}(\frac{u_3}{\sqrt{2}})\mathrm{erf}(u_1) + \mathrm{erf}(\frac{u_2}{\sqrt{2}})\mathrm{erf}(u_1)\mathrm{erf}(u_3)}{8\sqrt{\pi^3} - \mathrm{erf}(u_1)\mathrm{erf}(u_2)\mathrm{erf}(u_3)}$$

$$x_3(u_1, u_2, u_3) = u_3 + \sqrt{\pi}\,\mathrm{erf}\left(\frac{u_3}{\sqrt{2}}\right) \times \frac{4\pi\,\mathrm{erf}(\frac{u_2}{\sqrt{2}}) + 2\mathrm{erf}(\frac{u_1}{\sqrt{2}})\mathrm{erf}(u_2) + \mathrm{erf}(\frac{u_3}{\sqrt{2}})\mathrm{erf}(u_1)\mathrm{erf}(u_2)}{8\sqrt{\pi^3} - \mathrm{erf}(u_1)\mathrm{erf}(u_2)\mathrm{erf}(u_3)}$$

and the corresponding plot of the triply conjugate family of surfaces is shown in figure 6.

The plot of a generic surface with its conjugate net of coordinate lines is shown in figure 7.

The surface is asymptotically flat, and approaches to different planes depending on which quadrant we consider.

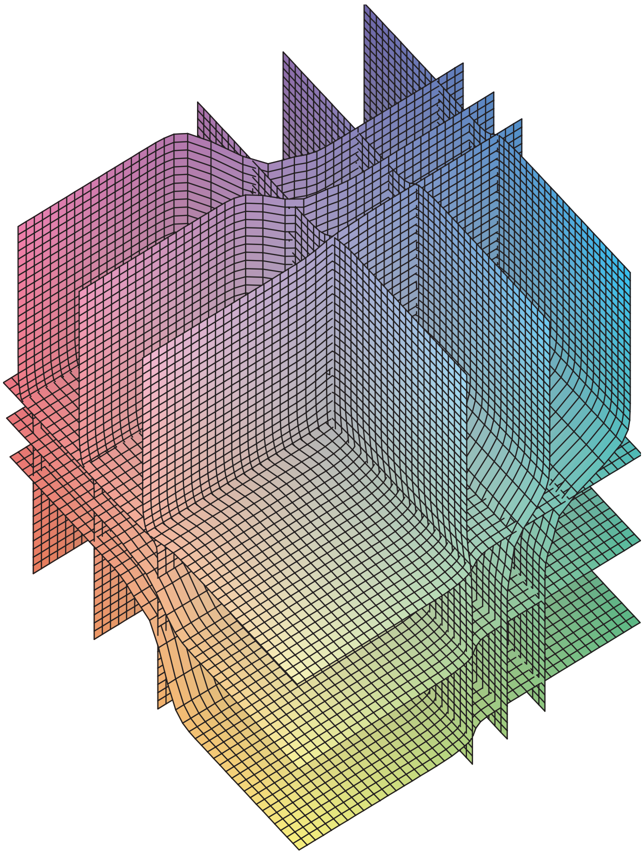

Figure 6. Plot of the triply conjugate family of surfaces for $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

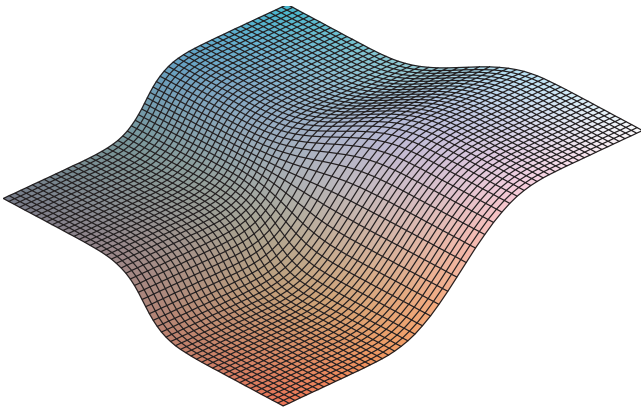

Figure 7. Plot of a generic surface with its conjugate net of coordinate lines for $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

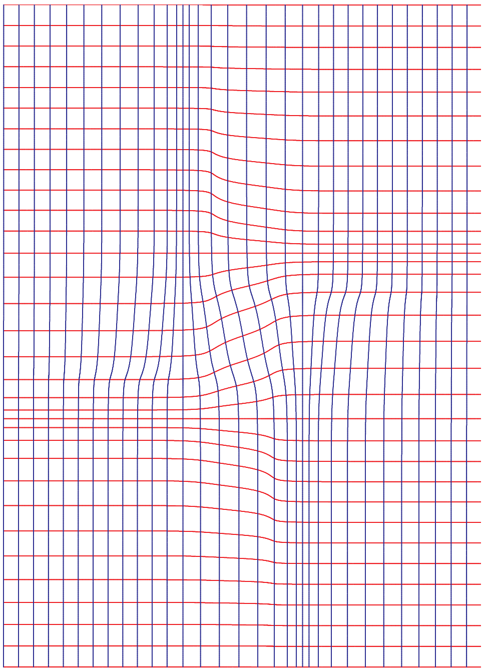

Figure 8. Plot for the bidimensional case $N = 2$ with $C = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $k = l = 1$.

• **Case (0, 3).** Now, we take

$$\mu_j(u_j) = \sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{u_j}{\sqrt{2}}\right) \qquad \nu_j(u_j) = (1 - u_j^2)\mathrm{e}^{-\frac{u_j^2}{2}}$$
$$\omega_j(u_j) = \tfrac{1}{2}(u_j^2 - 2)\mathrm{e}^{-u_j^2}.$$

For the bidimensional case $N = 2$ with $C = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $k = l = 1$ the net is given by

$$x_1(u_1, u_2) = u_1 + \sqrt{\frac{\pi}{32}}$$
$$\times \frac{(1 - u_1^2)\mathrm{e}^{-u_1^2}[-2\operatorname{erf}(\frac{u_2}{\sqrt{2}}) + (2 - (u_2^2 - 2)\mathrm{e}^{-u_2^2})\operatorname{erf}(\frac{u_1}{\sqrt{2}})]}{8 - 2(u_1^2 - 2)\mathrm{e}^{-u_1^2} - 2(u_2^2 - 2)\mathrm{e}^{-u_2^2} + (u_1^2 - 2)(u_2^2 - 2)\mathrm{e}^{-u_1^2 - u_2^2}}$$
$$x_2(u_1, u_2) = u_2 + \sqrt{\frac{\pi}{32}}$$
$$\times \frac{(1 - u_2^2)\mathrm{e}^{-u_2^2}[2\operatorname{erf}(\frac{u_1}{\sqrt{2}}) + (2 - (u_1^2 - 2)\mathrm{e}^{-u_1^2})\operatorname{erf}(\frac{u_2}{\sqrt{2}})]}{8 - 2(u_1^2 - 2)\mathrm{e}^{-u_1^2} - 2(u_2^2 - 2)\mathrm{e}^{-u_2^2} + (u_1^2 - 2)(u_2^2 - 2)\mathrm{e}^{-u_1^2 - u_2^2}}$$

as plotted in figure 8.

Unlike the error net, we see that in this case the bend of the coordinate lines decays asymptotically, and the net is a Cartesian net with a *Gaussian localized* perturbation.

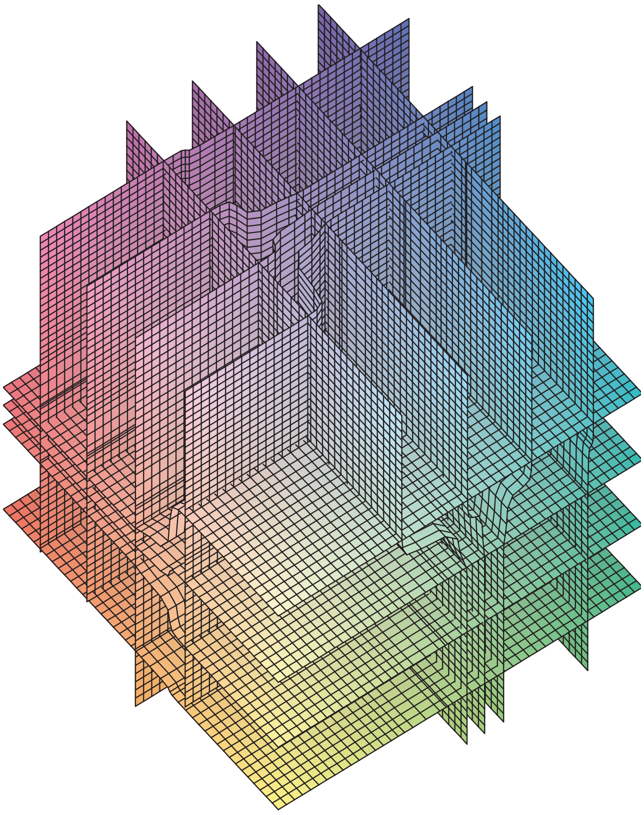

Figure 9. Plot for $N = 3$ with $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

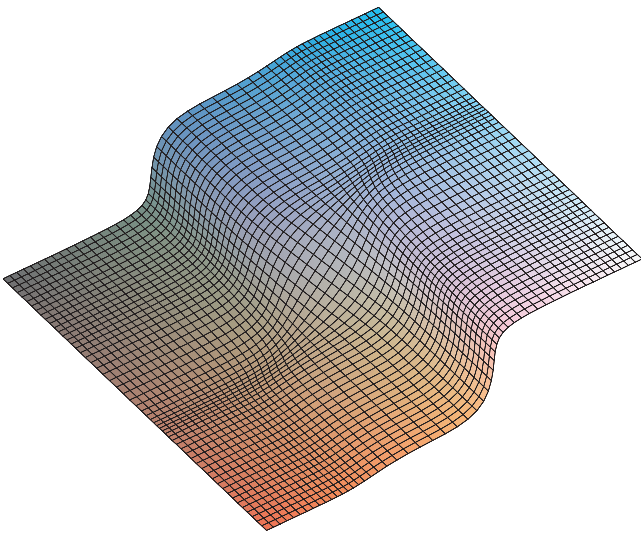

Figure 10. A generic coordinate surface for $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

For $N = 3$ and $C = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, the net is given by

$$
\begin{aligned}
x_1(u_1, u_2, u_3) = u_1 + \sqrt{\frac{\pi}{32}}\Bigg(\bigg((1-u_1^2)\mathrm{e}^{-u_1^2}\bigg[4\mathrm{erf}\bigg(\frac{u_3}{\sqrt{2}}\bigg) + \bigg(2\mathrm{erf}\bigg(\frac{u_2}{\sqrt{2}}\bigg) \\
+\mathrm{erf}\bigg(\frac{u_1}{\sqrt{2}}\bigg)(u_2^2-2)\mathrm{e}^{-u_2^2}\bigg)(u_3^2-2)\mathrm{e}^{-u_3^2}\bigg]\bigg) \\
\times\{8-(u_1^2-2)(u_2^2-2)(u_3^2-2)\mathrm{e}^{-u_1^2-u_2^2-u_3^2}\}^{-1}\Bigg) \\
x_2(u_1, u_2, u_3) = u_2 + \sqrt{\frac{\pi}{32}}\Bigg(\bigg((1-u_2^2)\mathrm{e}^{-u_2^2}\bigg[4\mathrm{erf}\bigg(\frac{u_1}{\sqrt{2}}\bigg) + \bigg(2\mathrm{erf}\bigg(\frac{u_3}{\sqrt{2}}\bigg) \\
+\mathrm{erf}\bigg(\frac{u_2}{\sqrt{2}}\bigg)(u_3^2-2)\mathrm{e}^{-u_3^2}\bigg)(u_1^2-2)\mathrm{e}^{-u_1^2}\bigg]\bigg) \\
\times\{8-(u_1^2-2)(u_2^2-2)(u_3^2-2)\mathrm{e}^{-u_1^2-u_2^2-u_3^2}\}^{-1}\Bigg) \\
x_3(u_1, u_2, u_3) = u_3 + \sqrt{\frac{\pi}{32}}\Bigg(\bigg((1-u_3^2)\mathrm{e}^{-u_3^2}\bigg[4\mathrm{erf}\bigg(\frac{u_2}{\sqrt{2}}\bigg) + \bigg(2\mathrm{erf}\bigg(\frac{u_1}{\sqrt{2}}\bigg) \\
+\mathrm{erf}\bigg(\frac{u_3}{\sqrt{2}}\bigg)(u_1^2-2)\mathrm{e}^{-u_1^2}\bigg)(u_2^2-2)\mathrm{e}^{-u_2^2}\bigg]\bigg) \\
\times\{8-(u_1^2-2)(u_2^2-2)(u_3^2-2)\mathrm{e}^{-u_1^2-u_2^2-u_3^2}\}^{-1}\Bigg)
\end{aligned}
$$

and the corresponding plot is shown in figure 9. A generic coordinate surface plots as in figure 10.

Acknowledgments

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